Heat Equation

conducting rod length L

heat can flow along the rod (x direction)

but not laterally (not "up" or "down")

temperature : $u(x,t)$ $\quad$ x: position $\quad$ t: time

the governing equation is the 1-D Heat Equation

$$\boxed{\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}}$$

$k > 0$ diffusivity constant

or $\quad u_t = k\, u_{xx}$

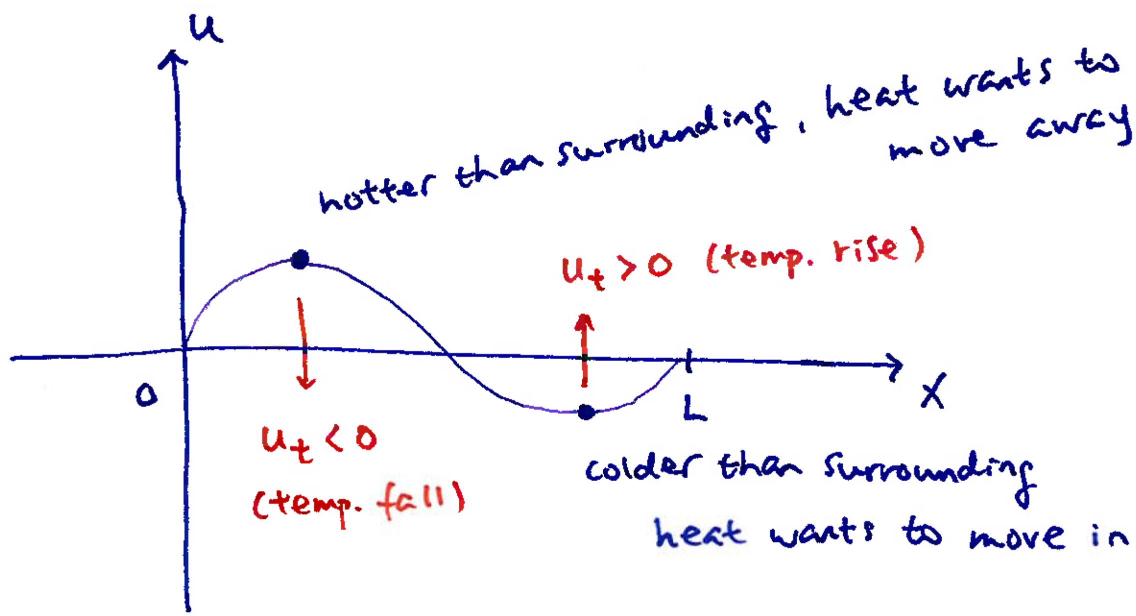
this is a partial diff. eq. (PDE)

the heat eq. comes from conservation of energy and Fourier's law what does $u_t = k u_{xx}$ say?

u_{xx} : concavity of u with respect to x

if $u_{xx} > 0$ (concave up) then $u_t > 0$ temp. increases

if $u_{xx} < 0$ (concave down) then $u_t < 0$ temp. decreases



How to solve $u_t = k u_{xx}$ $\quad 0<x<L, \quad t>0$?

set up: $u(0,t) = T_1$ (left end temp.)

$u(L,t) = T_2$ (right end temp.)

$u(x,0) = f(x)$ (initial temp. profile)

the first two are boundary conditions (BC)

the last one is an initial condition (IC)

we will solve the case where $T_1 = T_2 = 0$ (homogeneous BC's)

we will use the method of separation of variables

assume $u(x,t) = X(x)\,T(t)$ $\quad$ product of two functions

X : only of x

T : only of t

$$u_t = \frac{\partial}{\partial t}\Big(\underbrace{X(x)}_{\text{"constant"}}\,T(t)\Big) = X\,T'$$

likewise, $u_{xx} = X''\,T$

back to $u_t = k u_{xx}$

$$X T' = k X'' T$$

separate them: $\underbrace{\frac{X''}{X}}_{\text{only depends on } x} = \underbrace{\frac{T'}{kT}}_{\text{only depends on } t} = \text{constant} = -\lambda \quad (\lambda > 0)$

separation constant

$$\frac{X''}{X} = -\lambda \longrightarrow \boxed{X'' + \lambda X = 0}$$

$$\frac{T'}{kT} = -\lambda \longrightarrow \boxed{T' + k\lambda T = 0}$$

Two ODE's

BC's: $u(0,t) = 0 \rightarrow X(0)T(t) = 0 \rightarrow \boxed{X(0) = 0}$

$u(L,t) = 0 \rightarrow X(L)T(t) = 0 \rightarrow \boxed{X(L) = 0}$

solve $X'' + \lambda X = 0, \quad X(0) = X(L) = 0$

$$X(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$$

$$X(0) = 0 = A$$

$$X(L) = 0 = B\sin(\sqrt{\lambda}\,L)$$ require $B \neq 0$

$$\sin(\sqrt{\lambda}\,L) = 0$$

$$\sqrt{\lambda}\,L = n\pi \qquad n = 1, 2, 3, \ldots$$

$$\boxed{\lambda_n = \frac{n^2\pi^2}{L^2}}$$ eigenvalues

for each n, there is one X solution

$$\boxed{X_n(x) = \sin\left(\frac{n\pi}{L}x\right)}$$ (drop the scaling constant B)

eigenfunctions

now $T' + k\lambda T = 0$ use $\lambda = \frac{n^2\pi^2}{L^2}$

$$T' + \frac{kn^2\pi^2}{L^2} T = 0$$

$$T(t) = C e^{\left(-kn^2\pi^2/L^2\right)t}$$

for each $n = 1, 2, 3, \ldots$ there is one solution

$$\boxed{T_n(t) = e^{-kn^2\pi^2/L^2\, t}}$$

drop the scaling constant C

$$u(x,t) = X(x)\, T(t)$$

for each n, $u_n = X_n T_n$

the general solution is linear combo of all

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n e^{-kn^2\pi^2/L^2\, t} \sin\left(\frac{n\pi x}{L}\right)}$$

$c_n = ?$

one unused condition : $u(x,0) = f(x)$ (initial condition)

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$ sine series w/ half-period L

$$\boxed{c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$